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# Lie symmetries and particular solutions of Seiberg–Witten equations in $R^3$

**D V Aleynikov and E A Tolkachev**

B I Stepanov Institute of Physics, National Academy of Sciences of Belarus,
68 F Skaryna Avenue, 220072 Minsk, Belarus

E-mail: aleynik@dragon.bas-net.by and tea@dragon.bas-net.by

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**Abstract**
It is shown that the conformal group of three-dimensional Euclidean space $SO(4, 1)$ is a maximal group of Lie symmetries of the Seiberg–Witten and Freund's equations in $R^3$. Particular explicit solutions which are invariant under some subgroups of $SO(4, 1)$ are constructed.

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## 1. Introduction

The role of Seiberg–Witten equations [1] in both high-energy physics and topology is common knowledge. On one hand, the Seiberg–Witten equations describe the dynamics of twisted $N = 2$ supersymmetric Abelian gauge fields coupled to massless monopoles. The weak coupling limit of this theory is equivalent to the low-energy $N = 2$ super Yang–Mills theory in the strongly coupled region of field space [2]. On the other hand, a twisted version of supersymmetric theories is applicable to a classification of four-dimensional manifolds [3, 4], as well as three-dimensional ones [5, 6]. New topological invariants defined by moduli space of Seiberg–Witten equations were found [1, 4].

The particular solutions of Seiberg–Witten equations reduced to three-dimensional flat space $R^3$ were constructed in [7–9]. Furthermore, the Freund equations were proposed, which differ from Seiberg–Witten ones in the sign of the quadratic term [9].

This paper is devoted to a construction of group-invariant particular solutions of the Seiberg–Witten equations and those of Freund. We also include into consideration zero modes of the three-dimensional massless Dirac operator without magnetic fields (see, for instance, [10]).

  

## 2. Preliminaries

Consider the equations which include a two-component spinor field $\psi$ and a vector potential $\mathbf{A}$ in $R^3$:

$$\epsilon\psi^*\sigma_l\psi = H_l \qquad \sigma_l(\partial_l - \mathrm{i}A_l)\psi = 0 \qquad \varepsilon_{lkm}\partial_k A_m = H_l \tag{1}$$

where $\epsilon = \pm 1, 0$; $l, k, m = 1, 2, 3$; $\sigma_l$ are the Pauli matrices. For $\epsilon = 1$, these equations will be the Seiberg–Witten ones. In the case $\epsilon = -1$, we will think of the equations (1) as the Freund equations. For $\epsilon = 0$, we have the particular case of the zero-mode problem mentioned above.

The solutions of the second equation (1) are the zero modes of the three-dimensional massless Dirac operator. The general ansatz of these which depends upon a three-dimensional vector function was constructed in [11]:

$$\psi = \frac{1}{\sqrt{2(f + (\mathbf{fn}_3))}}\begin{pmatrix} f + (\mathbf{fn}_3) \\ (\mathbf{fn}_1) + \mathrm{i}(\mathbf{fn}_2)\end{pmatrix} \tag{2}$$

$$\mathbf{A} = \frac{[\nabla\mathbf{f}]}{2f} + \frac{f - (\mathbf{fn}_3)}{2f}\nabla\left(\arctan\frac{(\mathbf{fn}_2)}{(\mathbf{fn}_1)}\right) \qquad 0 = (\nabla\mathbf{f}) \tag{3}$$

where $f = \sqrt{\mathbf{f}^2}$; $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are arbitrary orthogonal unit vectors in $R^3$. Strictly speaking, we ought to define the functions $\psi$ and $\mathbf{A}$ on two overlapping regions of space. However, for our purpose, it is quite enough to deal with solutions (2) and (3) locally.

If we substitute the solution (2) into the first equation of (1) and the solution (3) into the third equation of (1), we obtain the second-order nonlinear differential equations in three independent and three dependent variables:

$$\Delta\mathbf{f} + \frac{\Gamma(\mathbf{f})}{f^2} + 2\epsilon f\mathbf{f} = 0 \qquad (\nabla\mathbf{f}) = 0 \tag{4}$$

where $\Gamma(\mathbf{f}) = [\nabla f^2[\nabla\mathbf{f}]] - (f_3[\nabla f_1\nabla f_2] - f_2[\nabla f_1\nabla f_3] + f_1[\nabla f_2\nabla f_3])$ (in short, we denote $f_1 = (\mathbf{fn}_1)$, $f_2 = (\mathbf{fn}_2)$ and $f_3 = (\mathbf{fn}_3)$); $\mathbf{r}$ is the independent vector and $\mathbf{f} = \mathbf{f}(\mathbf{r})$ is the dependent one.

## 3. Lie symmetries

One of the most useful methods for determining particular explicit solutions to partial differential equations (4) is to reduce them to ordinary differential equations which are invariant under some subgroups of the maximal Lie symmetry group of equations (4). For this purpose we need to construct the infinitesimal Lie algebra generators. Using the methods described in [12] gives (see appendix A)

$$\mathbf{L} = -[\mathbf{r}\nabla_\mathbf{r}] - [\mathbf{f}\nabla_\mathbf{f}] \tag{5}$$

$$\mathbf{P} = \nabla_\mathbf{r} \tag{6}$$

$$\mathbf{K} = 2\mathbf{r}(\mathbf{r}\nabla_\mathbf{r}) - \mathbf{r}^2\nabla_\mathbf{r} - 4\mathbf{r}(\mathbf{f}\nabla_\mathbf{f}) + 2[\mathbf{r}[\mathbf{f}\nabla_\mathbf{f}]] \tag{7}$$

$$D = (\mathbf{r}\nabla_\mathbf{r}) - 2(\mathbf{f}\nabla_\mathbf{f}). \tag{8}$$

Here

$$\nabla_\mathbf{r} = \mathbf{n}_1\frac{\partial}{\partial x} + \mathbf{n}_2\frac{\partial}{\partial y} + \mathbf{n}_3\frac{\partial}{\partial z} \qquad \nabla_\mathbf{f} = \mathbf{n}_1\frac{\partial}{\partial f_1} + \mathbf{n}_2\frac{\partial}{\partial f_2} + \mathbf{n}_3\frac{\partial}{\partial f_3}.$$

The commutation relations of generators (5)–(8) are

$$[L_i, K_j] = \varepsilon_{ijk} K_k \qquad [P_i, K_j] = 2(\delta_{ij} D + \varepsilon_{ijk} L_k)$$
$$[K_i, K_j] = [L_i, D] = 0 \qquad [K_i, D] = -K_i \qquad [P_i, D] = P_i.$$

So, we have Lie algebra of the $SO(4, 1)$ group (conformal group of three-dimensional Euclidean space). Generators **L**, **P**, **K** and $D$ determine infinitesimal rotations, shifts, inversions and dilatations, respectively. Note in particular that (5)–(8) generate fibre-preserving transformations, meaning that the transformations in **r** do not depend on the coordinates **f**.

The finite transformations of this group are

$$\text{rotations:} \quad \mathbf{r} \mapsto O\mathbf{r} \qquad \mathbf{f} \mapsto O\mathbf{f} \tag{9}$$

$$\text{shifts:} \quad \mathbf{r} \mapsto \mathbf{r} + \mathbf{b} \qquad \mathbf{f} \mapsto \mathbf{f} \tag{10}$$

$$\text{dilatations:} \quad \mathbf{r} \mapsto \exp(\lambda)\mathbf{r} \qquad \mathbf{f} \mapsto \exp(-2\lambda)\mathbf{f} \tag{11}$$

$$\text{inversions:} \quad \mathbf{r} \mapsto \frac{\mathbf{r} - \mathbf{r}^2\mathbf{v}}{1 - 2(\mathbf{rv}) + \mathbf{r}^2\mathbf{v}^2} \tag{12}$$

$$\mathbf{f} \mapsto (1 - 2(\mathbf{rv}) + \mathbf{r}^2\mathbf{v}^2)^2 \left( \mathbf{f} + \frac{2}{1 + \mathbf{q}^2}([\mathbf{qf}] + [\mathbf{q}[\mathbf{qf}]]) \right) \qquad \mathbf{q} = \frac{[\mathbf{vr}]}{1 - (\mathbf{vr})}$$

where $O$ is a three-dimensional orthogonal matrix; **b** and **v** are arbitrary vector parameters and $\lambda$ is a scalar parameter.

Suppose $\mathbf{f}(\mathbf{r})$ satisfies the equations (4), so does $\tilde{\mathbf{f}} \equiv T(g)\mathbf{f}(\mathbf{r})$, where $T(g)$ is an arbitrary combination of the finite transformations (9)–(12) of the $SO(4, 1)$ group. For pure rotations, shifts, dilatations and inversions we obtain, respectively,

$$\tilde{\mathbf{f}}_1 \equiv T(O)\mathbf{f}(\mathbf{r}) = O\mathbf{f}(O^{-1}\mathbf{r})$$
$$\tilde{\mathbf{f}}_2 \equiv T(\exp(\mathbf{bP}))\mathbf{f}(\mathbf{r}) = \mathbf{f}(\mathbf{r} - \mathbf{b})$$
$$\tilde{\mathbf{f}}_3 \equiv T(\exp(\lambda D))\mathbf{f}(\mathbf{r}) = \exp(-2\lambda)\mathbf{f}(\exp(-\lambda)\mathbf{r})$$
$$\tilde{\mathbf{f}}_4 \equiv T(\exp(\mathbf{vK}))\mathbf{f}(\mathbf{r}) = \frac{1}{(1 + 2(\mathbf{rv}) + \mathbf{r}^2\mathbf{v}^2)^2} \left( \mathbf{f}(\mathbf{r}') + \frac{2}{1 + \mathbf{q}'^2}([\mathbf{q}'\mathbf{f}(\mathbf{r}')] + [\mathbf{q}'[\mathbf{q}'\mathbf{f}(\mathbf{r}')]]) \right)$$

where

$$\mathbf{r}' = \frac{\mathbf{r} + \mathbf{r}^2\mathbf{v}}{1 + 2(\mathbf{rv}) + \mathbf{r}^2\mathbf{v}^2} \qquad \mathbf{q}' = \frac{[\mathbf{vr}]}{1 + (\mathbf{vr})}.$$

Thus, a class of solutions can be associated with the initial particular solution.

## 4. Group-invariant solutions

Consider, for instance, the solutions which are invariant under the $SO(3)$, $SO(2, 1)$ and $E(2)$ subgroups of $SO(4, 1)$. These subgroups can be generated by the following combinations of vector fields (5)–(8):

$$G_1 = (K_1 + \varsigma P_1)/2 \qquad G_2 = (K_2 + \varsigma P_2)/2 \qquad G_3 = L_3 \tag{13}$$

where $\varsigma = \pm 1, 0$. In the cases $\varsigma = 1$ and $\varsigma = -1$, vector fields (13) generate the $SO(3)$ and $SO(2, 1)$ groups, respectively. In the case $\varsigma = 0$, we have the $E(2)$ group.

Using the methods [13], we find the group-invariant ansatz in the form (see appendix B)

$$\mathbf{f} = \frac{\Omega(\rho_\varsigma)}{2(\mathbf{rn}_3)^3}(2(\mathbf{rn}_3)\mathbf{r} - (\mathbf{r}^2 + \varsigma)\mathbf{n}_3). \tag{14}$$

Here $\rho_\varsigma$ and $\Omega$ are group invariants

$$\rho_\varsigma = \frac{\mathbf{r}^2 + \varsigma}{2(\mathbf{r}\mathbf{n}_3)} \qquad \Omega = \frac{(\mathbf{r}\mathbf{n}_3)^2\sqrt{f^2 - (\mathbf{f}\mathbf{n}_3)^2}}{\sqrt{r^2 - (\mathbf{r}\mathbf{n}_3)^2}}.$$

The ansatz (14) reduces the equations (4) to the ordinary differential equations (see appendix B) for $\Omega(\rho_\varsigma)$ which are compatible if $\varsigma = \epsilon$. This implies that the Seiberg–Witten equations admit a $SO(3)$-invariant solution, but do not admit a solution which is invariant under the $SO(2,1)$ or $E(2)$ group. A $SO(2,1)$-invariant solution satisfies the Freund equations only. For the massless Dirac operator without magnetic fields, we have a solution which is invariant under the $E(2)$ group.

The reduced ordinary differential equations are easily solved and we get

$$\mathbf{f} = \pm\frac{2(\mathbf{r}\mathbf{n}_3)\mathbf{r} - (\mathbf{r}^2 + \epsilon)\mathbf{n}_3}{4(\mathbf{r}\mathbf{n}_3)^3\left(\rho_\epsilon^2 - \epsilon\right)^{3/2}}. \tag{15}$$

For $\epsilon = \varsigma = 0$ function (15) reproduces the special case of zero modes of the three-dimensional Dirac operator without magnetic field [10].

It should be noted that subgroups are determined up to inner automorphisms of the maximal symmetry group. So, a class of adjoint subgroups can be associated with a subgroup $H$: $\tilde{H} = gHg^{-1}$, where $g$ is an arbitrary element of the $SO(4,1)$ group.

Consider some adjoint subgroups to the $SO(3)$ group generated by vector fields (13) with $\varsigma = \epsilon = 1$. Let the inner automorphism be generated by the element $g = \exp\left(\frac{\pi}{2}(W_3 - L_3)\right)$, where $W_3 = (K_3 + P_3)/2$. Then

$$\exp\left(\frac{\pi}{2}(W_3 - L_3)\right) G_i \exp\left(-\frac{\pi}{2}(W_3 - L_3)\right) = L_i$$

where $i = 1, 2, 3$. We note that $[W_3, L_3] = 0$. So,

$$\exp\alpha(W_3 - L_3) = \exp(\alpha W_3)\exp(-\alpha L_3) = \exp(-\alpha L_3)\exp(\alpha W_3).$$

The one-parameter finite transformations $\exp(\alpha W_3)$ have the form

$$\begin{aligned} \mathbf{r} &\mapsto \frac{2\mathbf{r} - (2(1-\cos\alpha)(\mathbf{r}\mathbf{n}_3) + (\mathbf{r}^2 - 1)\sin\alpha)\mathbf{n}_3}{\mathbf{r}^2 + 1 - (\mathbf{r}^2 - 1)\cos\alpha - 2(\mathbf{r}\mathbf{n}_3)\sin\alpha} \\ \mathbf{f} &\mapsto \frac{(\mathbf{r}^2 + 1 - (\mathbf{r}^2 - 1)\cos\alpha - 2(\mathbf{r}\mathbf{n}_3)\sin\alpha)^2}{4}\left(\mathbf{f} + \frac{2}{1 + \mathbf{q}^2}([\mathbf{q}\mathbf{f}] + [\mathbf{q}[\mathbf{q}\mathbf{f}]])\right) \end{aligned} \tag{16}$$

where

$$\mathbf{q} = -\frac{(1-\cos\alpha)[\mathbf{n}_3\mathbf{r}]}{(1-\cos\alpha)(\mathbf{n}_3\mathbf{r}) - \sin\alpha}.$$

The solution which is invariant under the $SO(3)$ group generated by vector fields $\mathbf{L}$ can be obtained from (15):

$$\tilde{\mathbf{f}} = \pm T\left(\exp\left(\frac{\pi}{2}(W_3 - L_3)\right)\right)\frac{2(\mathbf{r}\mathbf{n}_3)\mathbf{r} - (\mathbf{r}^2 + 1)\mathbf{n}_3}{4(\mathbf{r}\mathbf{n}_3)^3\left(\rho_1^2 - 1\right)^{3/2}} = \mp\frac{\mathbf{r}}{2|\mathbf{r}|^3}. \tag{17}$$

So, we reproduce the well-known monopole-like solution [7, 8].

The next example is the $SO(3)$ group generated by vector fields

$$\exp(-\mathbf{v}\mathbf{K})\mathbf{L}\exp(\mathbf{v}\mathbf{K}) = \mathbf{L} + [\mathbf{v}\mathbf{K}].$$

In this case the group-invariant solution has the form

$$\tilde{\tilde{\mathbf{f}}} = T(\exp(\mathbf{v}\mathbf{K}))\tilde{\mathbf{f}} = T(\exp(\mathbf{v}\mathbf{K}))\frac{\mathbf{r}}{2|\mathbf{r}|^3} = \frac{\mathbf{r} + (\mathbf{v}\mathbf{r})\mathbf{r} + [[\mathbf{v}\mathbf{r}]\mathbf{r}]}{2|\mathbf{r} + (\mathbf{v}\mathbf{r})\mathbf{r} + [[\mathbf{v}\mathbf{r}]\mathbf{r}]|^3}. \tag{18}$$

Thus, the Seiberg–Witten equations admit a class of $SO(3)$-invariant solutions. In similar fashion we can obtain a class of $SO(2,1)$-invariant solutions satisfying the Freund equations as well as a class of $E(2)$-invariant solutions which satisfy the massless Dirac equation without magnetic fields.

We note in conclusion that solutions which are invariant under a two-parameter subgroups could be constructed. The most interesting example of these which satisfies the Seiberg–Witten equations was constructed in [14]:

$$\mathbf{f} = \frac{12}{(1+r^2)^3}(2[\mathbf{n}_3\mathbf{r}] + 2(\mathbf{n}_3\mathbf{r})\mathbf{r} + (1-r^2)\mathbf{n}_3).$$

This solution is invariant under a two-parameter Abelian torus group generated by vector fields $W_3$ and $L_3$. The group invariants are

$$\begin{aligned}
\rho &= \frac{\mathbf{r}^2+1}{2\sqrt{\mathbf{r}^2-(\mathbf{r}\mathbf{n}_3)^2}} \\
\Omega_1 &= \sqrt{\mathbf{r}^2-(\mathbf{r}\mathbf{n}_3)^2}(\mathbf{n}_3[\mathbf{r}\mathbf{f}]) \\
\Omega_2 &= \sqrt{\mathbf{f}^2}(\mathbf{r}^2-(\mathbf{r}\mathbf{n}_3)^2) \\
\Omega_3 &= (\mathbf{r}^2-1-2(\mathbf{r}\mathbf{n}_3)^2)(\mathbf{r}\mathbf{f}) + (\mathbf{r}^2+1)(\mathbf{r}\mathbf{n}_3)(\mathbf{f}\mathbf{n}_3).
\end{aligned} \tag{19}$$

Substitution of $\mathbf{f}$ into $\Omega_1$, $\Omega_2$ and $\Omega_3$ gives

$$\Omega_1 = \frac{3}{\rho^3} \qquad \Omega_2 = \frac{3}{\rho^2} \qquad \Omega_3 = 0.$$

However, we were not able to construct group-invariant solutions in general, because equations (4) reduce to the system of ordinary differential equations for the three unknown functions. This system requires separate analysis.

## 5. Conclusions

The conformal group of three-dimensional Euclidean space $SO(4,1)$ is a maximal group of Lie symmetries of the Seiberg–Witten and Freund equations in $R^3$. However, the Seiberg–Witten equations admit a $SO(3)$-invariant solution, while the $SO(2,1)$-invariant solutions satisfy the Freund equations only. For massless Dirac operator without magnetic fields, we have a solution which is invariant under $E(2)$ group.

## Appendix A

The infinitesimal Lie group transformations have the form [12]

$$\mathbf{v} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z} + \varphi\frac{\partial}{\partial f_1} + \psi\frac{\partial}{\partial f_2} + \chi\frac{\partial}{\partial f_3} \tag{A.1}$$

where $\xi$, $\eta$, $\zeta$, $\varphi$, $\psi$ and $\chi$ depend on $\mathbf{r}$, $\mathbf{f}$.

The second prolongation $\mathbf{pr}^{(2)}\mathbf{v}$ of vector field (A.1) is

$$\begin{aligned}
\mathbf{pr}^{(2)}\mathbf{v} = \mathbf{v} &+ \varphi^x\frac{\partial}{\partial f_{1x}} + \varphi^y\frac{\partial}{\partial f_{1y}} + \varphi^y\frac{\partial}{\partial f_{1z}} + \psi^x\frac{\partial}{\partial f_{2x}} + \psi^y\frac{\partial}{\partial f_{2y}} \\
&+ \psi^y\frac{\partial}{\partial f_{2z}} + \chi^x\frac{\partial}{\partial f_{3x}} + \chi^y\frac{\partial}{\partial f_{3y}} + \chi^y\frac{\partial}{\partial f_{3z}} + \varphi^{xx}\frac{\partial}{\partial f_{1xx}} + \varphi^{xy}\frac{\partial}{\partial f_{1xy}} \\
&+ \varphi^{xz}\frac{\partial}{\partial f_{1xz}} + \varphi^{yy}\frac{\partial}{\partial f_{1yy}} + \varphi^{yz}\frac{\partial}{\partial f_{1yz}} + \varphi^{zz}\frac{\partial}{\partial f_{1zz}} + \psi^{xx}\frac{\partial}{\partial f_{2xx}}
\end{aligned}$$

$$+\psi^{xy}\frac{\partial}{\partial f_{2xy}}+\psi^{xz}\frac{\partial}{\partial f_{2xz}}+\psi^{yy}\frac{\partial}{\partial f_{2yy}}+\psi^{yz}\frac{\partial}{\partial f_{2yz}}+\psi^{zz}\frac{\partial}{\partial f_{2zz}}$$
$$+\chi^{xx}\frac{\partial}{\partial f_{3xx}}+\chi^{xy}\frac{\partial}{\partial f_{3xy}}+\chi^{xz}\frac{\partial}{\partial f_{3xz}}+\chi^{yy}\frac{\partial}{\partial f_{3yy}}+\chi^{yz}\frac{\partial}{\partial f_{3yz}}+\chi^{zz}\frac{\partial}{\partial f_{3zz}}$$

where

$$\begin{aligned}
\varphi^x &= D_x(\varphi-\xi f_{1x}-\eta f_{1y}-\zeta f_{1z})+\xi f_{1xx}+\eta f_{1xy}+\zeta f_{1xz}\\
&= \varphi_x+(\varphi_{f_1}-\xi_x)f_{1x}+\varphi_{f_2}f_{2x}+\varphi_{f_3}f_{3x}-\xi_{f_1}(f_{1x})^2\\
&\quad -\xi_{f_2}f_{1x}f_{2x}-\xi_{f_3}f_{1x}f_{3x}-\eta_x f_{1y}-\eta_{f_1}f_{1x}f_{1y}-\eta_{f_2}f_{2x}f_{1y}\\
&\quad -\eta_{f_3}f_{3x}f_{1y}-\zeta_x f_{1z}-\zeta_{f_1}f_{1x}f_{1z}-\zeta_{f_2}f_{2x}f_{1z}-\zeta_{f_3}f_{3x}f_{1z}\\
\varphi^y &= D_y(\varphi-\xi f_{1x}-\eta f_{1y}-\zeta f_{1z})+\xi f_{1xy}+\eta f_{1yy}+\zeta f_{1yz}\\
\varphi^z &= D_z(\varphi-\xi f_{1x}-\eta f_{1y}-\zeta f_{1z})+\xi f_{1xz}+\eta f_{1yz}+\zeta f_{1zz}\\
\psi^x &= D_x(\psi-\xi f_{2x}-\eta f_{2y}-\zeta f_{2z})+\xi f_{2xx}+\eta f_{2xy}+\zeta f_{2xz}\\
&\quad\cdots\cdots\\
\chi^z &= D_z(\chi-\xi f_{3x}-\eta f_{3y}-\zeta f_{3z})+\xi f_{3xz}+\eta f_{3yz}+\zeta f_{3zz}\\
\varphi^{xx} &= D_{xx}(\varphi-\xi f_{1x}-\eta f_{1y}-\zeta f_{1z})+\xi f_{1xxx}+\eta f_{1xxy}+\zeta f_{1xxz}\\
\varphi^{xy} &= D_{xy}(\varphi-\xi f_{1x}-\eta f_{1y}-\zeta f_{1z})+\xi f_{1xxy}+\eta f_{1xyy}+\zeta f_{1xyz}\\
&\quad\cdots\cdots\\
\psi^{xy} &= D_{xy}(\psi-\xi f_{2x}-\eta f_{2y}-\zeta f_{2z})+\xi f_{2xxy}+\eta f_{2xyy}+\zeta f_{2xyz}\\
&\quad\cdots\cdots\\
\chi^{zz} &= D_{zz}(\chi-\xi f_{3x}-\eta f_{3y}-\zeta f_{3z})+\xi f_{3xzz}+\eta f_{3yzz}+\zeta f_{3zzz}.
\end{aligned}$$

Here $D$ is the total derivative operator:

$$D_xP(\mathbf{r},\mathbf{f})=\frac{\partial P}{\partial x}+\frac{\partial f_1}{\partial x}\frac{\partial P}{\partial f_1}+\frac{\partial f_2}{\partial x}\frac{\partial P}{\partial f_2}+\frac{\partial f_3}{\partial x}\frac{\partial P}{\partial f_3}=P_x+f_{1x}P_{f_1}+f_{2x}P_{f_2}+f_{3x}P_{f_3}$$

for arbitrary function $P(\mathbf{r},\mathbf{f})$.

It is convenient to denote the system of equations (4) by

$$R(\mathbf{f},f_{ix},f_{iy},f_{iz},f_{ixx},f_{ixy},\ldots.f_{iyz},f_{izz})=0.$$

This system is invariant under infinitesimal transformation (A.1) if

$$\mathbf{pr}^{(2)}\mathbf{v}[R(\mathbf{f},f_{ix},f_{iy},f_{iz},f_{ixx},f_{ixy},\ldots,f_{iyz},f_{izz})]=0 \tag{A.2}$$

for all solution of (4) ($i=1,2,3$) [12]. The infinitesimal condition (A.2) leads to the differential constraints

$$\begin{aligned}
&\xi_{f_i}=\eta_{f_i}=\zeta_{f_i}=\varphi_{f_if_j}=\psi_{f_if_j}=\chi_{f_if_j}=0\\
&\varphi_{f_ixx}=\varphi_{f_ixy}=\cdots=\varphi_{f_izz}=0\\
&\psi_{f_ixx}=\psi_{f_ixy}=\cdots=\psi_{f_izz}=0\\
&\chi_{f_ixx}=\chi_{f_ixy}=\cdots=\chi_{f_izz}=0\\
&\varphi_{f_1}=\psi_{f_2}=\chi_{f_3}=\frac{\varphi f_1+\psi f_2+\chi f_3}{f_1^2+f_2^2+f_3^2}
\end{aligned}$$

$$\begin{aligned}
&\xi_x=\eta_y=\zeta_z\quad \varphi_{f_2}=\xi_y\quad \varphi_{f_3}=\xi_z\quad \psi_{f_1}=\eta_x\quad \psi_{f_3}=\eta_z\quad \chi_{f_1}=\zeta_x\quad \chi_{f_2}=\zeta_y\\
&\xi_y=-\eta_x\quad \xi_y=-\zeta_x\quad \eta_z=-\zeta_y\quad \varphi_{f_1x}=-2\xi_{xx}\quad \varphi_{f_1y}=-2\eta_{yy}\quad \varphi_{f_1z}=-2\zeta_{zz}\\
&\varphi_{f_1x}=2\varphi_{f_2y}=2\varphi_{f_3z}\quad \psi_{f_2y}=2\psi_{f_1x}=2\psi_{f_3z}\quad \chi_{f_3z}=2\chi_{f_1x}=2\chi_{f_2y}\quad \psi_{f_3x}=\varphi_{f_2z}\\
&=\chi_{f_1y}=0\quad \varphi_{f_1}=-2\xi_x.
\end{aligned}$$

The functions which satisfy these constraints are

$$\begin{aligned}
\xi &= a_1(z^2+y^2-x^2)/2 - a_2xy - a_3xz - a_4x + a_5y + a_6z + a_8\\
\eta &= a_2(z^2+x^2-y^2)/2 - a_1xy - a_3yz - a_5x - a_4y + a_7z + a_9\\
\zeta &= a_3(x^2+y^2-z^2)/2 - a_1xz - a_2yz - a_6x - a_7y - a_4z + a_{10}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\varphi &= a_1(2xf_1+yf_2+zf_3) + a_2(2yf_1 - xf_2) + a_3(2zf_1 - xf_3) + 2a_4f_1 + a_5f_2 + a_6f_3\\
\psi &= a_2(xf_1+2yf_2+zf_3) + a_1(2xf_2 - yf_1) + a_3(2zf_2 - yf_3) + 2a_4f_2 - a_5f_1 + a_7f_3\\
\chi &= a_3(xf_1+yf_2+2zf_3) + a_1(2xf_3 - zf_1) + a_2(2yf_3 - zf_2) + 2a_4f_3 - a_6f_1 - a_7f_2.
\end{aligned} \tag{A.4}$$

Substituting (A.3) and (A.4) into (A.1) gives

$$\mathbf{v} = -a_1/2K_1 - a_2/2K_2 - a_3/2K_3 - a_4D + a_5L_3 - a_6L_2 + a_7L_1 + a_8P_1 + a_9P_2 + a_{10}P_3.$$

## Appendix B

The vector fields (13) are

$$\begin{aligned}
G_1 = \frac{x^2-y^2-z^2+\varsigma}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} - (2xf_1+yf_2+zf_3)\frac{\partial}{\partial f_1}\\
+ (yf_1 - 2xf_2)\frac{\partial}{\partial f_2} + (zf_1 - 2xf_3)\frac{\partial}{\partial f_3}
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
G_2 = xy\frac{\partial}{\partial x} + \frac{y^2-x^2-z^2+\varsigma}{2}\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} + (xf_2 - 2yf_1)\frac{\partial}{\partial f_1}\\
- (xf_1+2yf_2+zf_3)\frac{\partial}{\partial f_2} + (zf_2 - 2yf_3)\frac{\partial}{\partial f_3}
\end{aligned} \tag{B.2}$$

$$G_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + f_2\frac{\partial}{\partial f_1} - f_1\frac{\partial}{\partial f_2}. \tag{B.3}$$

Consider the matrices

$$M = \begin{pmatrix} \frac{x^2-y^2-z^2+\varsigma}{2} & xy & xz\\ xy & \frac{y^2-x^2-z^2+\varsigma}{2} & yz\\ y & -x & 0 \end{pmatrix}$$

and $\widetilde{M} =$

$$\begin{pmatrix} \frac{x^2-y^2-z^2+\varsigma}{2} & xy & xz & -(2xf_1+yf_2+zf_3) & yf_1-2xf_2 & zf_1-2xf_3\\ xy & \frac{y^2-x^2-z^2+\varsigma}{2} & yz & xf_2-2yf_1 & -(xf_1+2yf_2+zf_3) & zf_2-2yf_3\\ y & -x & 0 & f_2 & -f_1 & 0 \end{pmatrix}.$$

The solutions which are invariant under group generated by vector fields (B.1)–(B.3) exist if

$$\operatorname{rank} M = \operatorname{rank}\widetilde{M} < 3. \tag{B.4}$$

Condition (B.4) is equvalent to

$$xf_2 = yf_1 \tag{B.5}$$

$$2xzf_3 = (z^2-x^2-y^2-\varsigma)f_1. \tag{B.6}$$

Let $x = r_\perp \cos\theta$, $y = r_\perp \sin\theta$, $z$, $f_1 = f \sin\theta_1 \cos(\theta+\gamma)$, $f_2 = f \sin\theta_1 \sin(\theta+\gamma)$, $f_3 = f \cos\theta_1$. Then

$$\frac{\partial}{\partial x} = -\frac{\sin\theta}{r_\perp}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r_\perp} + \frac{\sin\theta}{r_\perp}\frac{\partial}{\partial\gamma}$$

$$\frac{\partial}{\partial y} = \frac{\cos\theta}{r_\perp}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r_\perp} - \frac{\cos\theta}{r_\perp}\frac{\partial}{\partial\gamma}$$

$$\frac{\partial}{\partial f_1} = \cos(\theta+\gamma)\left(\sin\theta_1\frac{\partial}{\partial f} + \frac{\cos\theta_1}{f}\frac{\partial}{\partial\theta_1}\right) - \frac{\sin(\theta+\gamma)}{f\sin\theta_1}\frac{\partial}{\partial\gamma}$$

$$\frac{\partial}{\partial f_2} = \sin(\theta+\gamma)\left(\sin\theta_1\frac{\partial}{\partial f} + \frac{\cos\theta_1}{f}\frac{\partial}{\partial\theta_1}\right) + \frac{\cos(\theta+\gamma)}{f\sin\theta_1}\frac{\partial}{\partial\gamma}$$

$$\frac{\partial}{\partial f_3} = \cos\theta_1\frac{\partial}{\partial f} - \frac{\sin\theta_1}{f}\frac{\partial}{\partial\theta_1}.$$

Rewrite (B.1)–(B.3) in new coordinates,

$$G_1 = \frac{r_\perp^2 + z^2 - \varsigma}{2r_\perp}\sin\theta\frac{\partial}{\partial\theta} + \frac{r_\perp^2 - z^2 + \varsigma}{2}\cos\theta\frac{\partial}{\partial r_\perp} + r_\perp z\cos\theta\frac{\partial}{\partial z} + \left(\frac{r_\perp^2 - z^2 + \varsigma}{2r_\perp}\sin\theta + z\cot\theta_1\sin(\theta+\gamma)\right)\frac{\partial}{\partial\gamma} - 2r_\perp f\cos\theta\frac{\partial}{\partial f} - z\cos(\theta+\gamma)\frac{\partial}{\partial\theta_1} \qquad \text{(B.7)}$$

$$G_2 = -\frac{r_\perp^2 + z^2 - \varsigma}{2r_\perp}\cos\theta\frac{\partial}{\partial\theta} + \frac{r_\perp^2 - z^2 + \varsigma}{2}\sin\theta\frac{\partial}{\partial r_\perp} + r_\perp z\sin\theta\frac{\partial}{\partial z} - \left(\frac{r_\perp^2 - z^2 + \varsigma}{2r_\perp}\cos\theta + z\cot\theta_1\cos(\theta+\gamma)\right)\frac{\partial}{\partial\gamma} - 2r_\perp f\sin\theta\frac{\partial}{\partial f} - z\sin(\theta+\gamma)\frac{\partial}{\partial\theta_1} \qquad \text{(B.8)}$$

$$G_3 = -\frac{\partial}{\partial\theta}. \qquad \text{(B.9)}$$

Equations (B.5), (B.6) in these coordinates are

$$\sin\gamma = 0 \quad (\gamma = 0) \qquad \text{(B.10)}$$

$$2r_\perp z\cos\theta_1 = \left(z^2 - r_\perp^2 - \varsigma\right)\sin\theta_1. \qquad \text{(B.11)}$$

The characteristic equations

$$\frac{dz}{2r_\perp z} = \frac{dr_\perp}{r_\perp^2 - z^2 + \varsigma} \qquad \text{and} \qquad \frac{dz}{z} = -\frac{df}{2f}$$

give two group invariants

$$\frac{r_\perp^2 + z^2 + \varsigma}{2z} = C_1 \qquad \text{and} \qquad fz^2 = C_2$$

respectively. Using (B.10) and (B.11) gives

$$f_1 = f \sin\theta_1 \cos\theta = \frac{xC_2}{z^2\sqrt{C_1^2 - \varsigma}}$$

$$f_2 = f \sin\theta_1 \sin\theta = \frac{yC_2}{z^2\sqrt{C_1^2 - \varsigma}}$$

$$f_3 = f \cos\theta_1 = \frac{\left(z^2 - r_\perp^2 - \varsigma\right) C_2}{2z^3\sqrt{C_1^2 - \varsigma}}$$

or

$$f_1 = \frac{\Omega x}{z^2} \qquad f_2 = \frac{\Omega y}{z^2} \qquad f_3 = \frac{\Omega(z^2 - x^2 - y^2 - \varsigma)}{2z^3}$$

where

$$\rho \equiv C_1 \qquad \Omega \equiv \frac{C_2}{\sqrt{C_1^2 - \varsigma}}.$$

We note that $\Omega > 0$. Taking into account that

$$\frac{\partial f_1}{\partial x} = \frac{x^2}{z^3}\frac{\partial\Omega}{\partial\rho} + \frac{\Omega}{z^2} \qquad \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = \frac{xy}{z^3}\frac{\partial\Omega}{\partial\rho} \qquad \frac{\partial f_1}{\partial z} = \frac{x(z-\rho)}{z^3}\frac{\partial\Omega}{\partial\rho} - \frac{2x\Omega}{z^3}$$

$$\frac{\partial f_2}{\partial y} = \frac{y^2}{z^3}\frac{\partial\Omega}{\partial\rho} + \frac{\Omega}{z^2} \qquad \frac{\partial f_2}{\partial z} = \frac{y(z-\rho)}{z^3}\frac{\partial\Omega}{\partial\rho} - \frac{2y\Omega}{z^3} \qquad \frac{\partial f_3}{\partial x} = \frac{x(z-\rho)}{z^3}\frac{\partial\Omega}{\partial\rho} - \frac{x\Omega}{z^3}$$

$$\frac{\partial f_3}{\partial y} = \frac{y(z-\rho)}{z^3}\frac{\partial\Omega}{\partial\rho} - \frac{y\Omega}{z^3} \qquad \frac{\partial f_3}{\partial z} = \frac{(z-\rho)^2}{z^3}\frac{\partial\Omega}{\partial\rho} + \frac{(3\rho - 2z)\Omega}{z^3}$$

Equations (4) reduce to the ordinary differential equations

$$(\rho^2 - \varsigma)\frac{\partial^2\Omega}{\partial\rho^2} + 5\rho\frac{\partial\Omega}{\partial\rho} + 3\Omega - \frac{\Omega\varsigma}{\rho^2 - \varsigma} + 2\epsilon\Omega^2\sqrt{\rho^2 - \varsigma} = 0 \tag{B.12}$$

$$(\rho^2 - \varsigma)\frac{\partial\Omega}{\partial\rho} + 3\rho\Omega = 0. \tag{B.13}$$

Equation (B.13) admits the solution

$$\Omega = \frac{C}{2\sqrt{(\rho^2 - \varsigma)^3}} \tag{B.14}$$

where $C$ is a positive real constant. Substituting (B.14) into (B.12) gives $C = 1$ and $\varsigma = \epsilon$.

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